Mechanism Design With Budget Constraints and a Population of Agents

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Abstract

This paper finds welfare- and revenue-maximizing mechanisms for assigning a divisible good to a population of budget-constrained agents who have independently distributed private valuations and budgets. Both optimal mechanisms feature a linear price for the good. The welfare-maximizing mechanism additionally has a uniform lump-sum transfer to all agents and a higher linear price than the revenue-maximizing mechanism. This transfer increases welfare because it ameliorates the key difficulty in the aforementioned setting: agents with high valuations cannot purchase an efficient amount of the good due to their budget constraints. Two extensions are considered: a relaxation of independence between valuations and budgets and the introduction of aggregate uncertainty.

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1 Introduction

What is the optimal mechanism for allocating a good to a population with private valuations and private budget constraints? Similar questions have been considered in several recent papers, but in all of them until this point, there has either been a finite number of agents or a finite discrete set of types. The innovation of this paper is that it considers the problem with a continuum of agents and types and here the optimal mechanisms derived are uniquely simple, linear prices with lump-sum transfers.

To illustrate the problem and its solution, consider a finite example: Suppose that there are two agents with unit valuations v_1, v_2 and a common budget constraint w such that $w < v_1 < 2w < v_2$. To maximize welfare, the principal would like to sell the good to agent 2. However, agent 2 can only afford the price w, and at that price the principal faces the problem that agent 1 would also like to purchase the good. Since the good is divisible, one possible mechanism is for the principal to sell one half of the good to each agent for the price $w/2$. While this is incentive compatible, the principal can do better and achieve the first-best by giving each agent w and selling the good at a price of $2w$. Both agents can afford this price because of the transfer, but only agent 2 will want to purchase the good at this price. Moreover, this mechanism produces a balanced budget as each agent receives w and the high-valuation agent pays $2w$. An alternative implementation would be for the principal to give each agent half a unit of the good and then allowing agent 2 to buy agent 1's allocation for the price w, thus achieving the first-best outcome as above.

While the above example differs from the setting considered here, it illustrates a key principle, namely that transfers to agents can be used to weaken budget constraints and improve welfare. In this paper, an agent's utility is linear in both the quantity of the good and money while the agents' valuations and budget constraints are independently distributed and private. The principal has a finite supply of the good and must satisfy a weak balanced budget constraint. In both the welfare- and revenue-maximizing settings, the optimal mechanisms feature a linear price p for the good. In line with the previous example, the utilitarian welfare-maximizing mechanism also features a uniform lump-sum transfer from the principal to the agents, whereas the revenue-maximizing mechanism does not.[1] The principal can implement

[1]This is in line with Pai and Vohra (2014) who find that a revenue-maximizing principal does

this mechanism via cash transfers or alternatively in-kind transfers of the good and then allow resale, which is a *prior-free* implementation.[2] The principal's transfers to the agents are uniform because types are unobservable, and therefore high-value or low-wealth agents cannot be profitably targeted to receive higher transfers.

Economic situations where a principal wishes to distribute a divisible good to budget-constrained agents abound. In the case of welfare maximization, such settings may be: the provision of healthcare or education in a government regulated system or the privatization of a government-owned enterprise. In each of these settings, budget constraints may be significant and may stand in the way of an efficient allocation of resources.[3] For the case of revenue maximization, examples include: a monopolist facing a budget-constrained population, the privatization of companies, and the sale of government land or the sale of bonds. In particular, the uniform auction method of selling bonds is a dominant strategy implementation of the optimal mechanism derived in this paper. Furthermore, if a principal wishes to maximize societal welfare from some larger planning problem, there may be a preference to maximize revenue or a joint welfare/revenue objective when selling the good so that the revenue generated can be applied to other welfare-improving endeavors. The proof of the main theorem also establishes that linear mechanisms are optimal for joint objectives.

The structure of the paper is as follows: First, I discuss related literature. In section 2, I introduce the framework and the essential assumptions. Section 3 contains the necessary lemmas and the main result which demonstrates the optimality of a linear pricing function in achieving utilitarian efficiency or revenue maximization. In section 4, I consider a relaxation of the independence assumption and the case of aggregate uncertainty. Section 5 concludes. The appendix includes all proofs and an example of how the theorem fails when the given assumptions do not hold. Finally,

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not make transfers to the agents in finite auction setting.

[2]See Devanur et al. (2013) and Hartline (2013) for discussions of approximately optimal mechanisms without priors.

[3]For example, in the mass privatization auctions in Central and Eastern Europe, vouchers were distributed which could then be used for bidding for different companies. There was significant variations in implementation across countries and in some countries, including Russia, resale of these vouchers was permitted (Boycko et al. (1994)). Furthermore, budget constraints were significant in that setting, Estrin (1991) states that the total sum of private savings (330 billion crowns) was 10% of the value of companies (3.3 billion crowns) being privatized in Czechoslovakia and 1% in Poland (78 million zloty / 64 billion zloty). For a more thorough overview, see Border (1991) and Boycko et al. (1996).

an online supplement covers dominant strategy implementations, large approximately optimal mechanisms, and production.

Related Literature

The addition of budget constraints to standard mechanism design problems has been studied widely in the literature. For example, Che and Gale (1998) showed that the standard revenue equivalence results of Myerson (1981) do not hold when agents have budget constraints. Specifically, they find that in a standard auction setting, the first price auction outperforms the second price auction because agents hedge their bids in a first-price setting and therefore budget constraints are less binding. Burkett (2015) shows that this performance gap can be eliminated if the principal optimally constrains the agents' budgets. Other work that also takes place in a finite auction setting include: a demonstration of a possible failure of the linkage principle (Fang and Parreiras (2002)), equilibrium analysis in first-price auctions (theoretically, Kotowski (2013) and experimentally Kotowski and Li (2010)), affiliated second-price auctions (Fang and Parreiras (2003)), welfare-maximization with externalities in a school design setting (Mestieri (2010)), and a setting without quasilinear utility where Baisa (2016) demonstrated the superior performance of probabilistic mechanisms. In general, I take a divisible good interpretation, but an indivisible good interpretation is possible as well with the good being distributed probabilistically.

The case of a common public budget constraint has been considered by Laffont and Robert (1996) and Maskin (2000) who have solved for revenue-maximizing and welfare-maximizing mechanisms, respectively.

When budget constraints are individual and unobservable, agents' types become two-dimensional. The multi-dimensional mechanism design literature is large and features two common approaches: one is to assume that each of the agent's two characteristics can take on one of two different values and the problem becomes one of finite parameter programming (see Armstrong and Rochet (1999)), while the other is to show that there is a one-dimensional formulation and solve that problem (for example, see Armstrong (1996), Jehiel et al. (1996), and Che and Gale (2000)). In Armstrong (1996), agents have homogenous preferences specified by a vector in $\mathbb{R}^k_+$ and therefore their preferences can be expressed by a direction and a radius. He solves for the optimal mechanism along each direction and in certain cases, it happens to be everywhere incentive compatible and therefore optimal. Here instead, I start with any given mechanism and improve it along each budget level. This yields a class of (take-it-or-leave-it) mechanisms and I find the optimal one in such a class. This solution may not be admissible in the original problem, but I then smooth it in such a way that it becomes so and it is therefore a solution to the overall mechanism design problem. The solution turns out to be linear pricing, possibly with lump-sum transfers.

There are four papers in the literature on mechanism design with budget constraints that are most closely related to the present work. The first is Che and Gale (2000) who consider revenue maximization with one buyer. They find a convex pricing function in terms of probabilities to be revenue maximizing. While the objective of their mechanism is the same as in this paper, the difference in the found optimal mechanisms is due to a difference in single-agent versus continuum constraints.

The other three closely related papers are Pai and Vohra (2014) and Che, Gale, Kim (2013a, 2013b). The first focuses on a single-good auction with a finite number of bidders. The fact that a single good is being sold rather than a supply differentiates that model from the current one. Such an optimal design problem degenerates as the number of agents grows because with high probability there exists an agent with both a high valuation and a high budget. The latter two papers focus on welfare maximization with a continuum of agents. A key difference though is that they generally focus on a $2x2$ type setting, that is two possible valuations and two possible budgets. The optimal mechanisms found there are quite different featuring a system of taxes and subsidies. The simplicity of the current model's optimal solutions is reminiscent of Azevedo and Leshno (2016) who find that a supply and demand framework applies to a continuum matching setting, unlike the standard finite setting.

2 Framework

2.1 Setup

There is a single good to be distributed by a principal with finite aggregate supply S. There is a unit measure of agents, each of whom is defined by two attributes, wealth, w, and value, v. Agents are risk-neutral with linear utility in the quantity of the good and money. An agent's per-unit value for the good is determined by her value type, v. If an agent has a quantity of the good x and money m, then her utility is $U(v, x, m) = xv + m$. An agent's consumption of money is nonnegative, which constitutes her budget constraint.

Agents' attributes are distributed according to two independent distributions, F for values and G for budgets, each with continuously differentiable densities on a bounded non-negative support. Thus, knowing an agent's budget or value type reveals no information about the other attribute. So there is no incentive to favor high- or low-budget agents from a purely correlation point of view.

For example, in the provision of healthcare, wealth and health may be independent. In other cases, such as the distribution of food or housing allowances, agents' budgets and values may be negatively correlated. Finally, in some cases, such as the "license raj" in India where the government allocated production quotas to family firms, some firms may be more efficient than others (see Esteban and Ray (2006)). In those cases, wealthy firms may have a higher valuation of production because they are more efficient, which is how they became wealthy in the first place. The main theorem will focus on the case of independent distributions of agents' types and section 4.1 considers a relaxation by allowing for a type of positive correlation.

Note: The distributions F, G define the aggregate makeup of the population. The formulation here has no aggregate uncertainty and integrals are with respect to an aggregation over all agents rather than taking an expectation with respect to some underlying uncertainty. Aggregate uncertainty is considered in section 4.2.

I consider two different mechanism design problems. In the first, a principal (perhaps the government or some other public institution) wishes to maximize the welfare of the agents. To achieve this, he wishes to assign the good to the agents with the highest valuation of the good. The mechanism design problem is to find assignment and transfer rules $x, t : V \times W \to \mathbb{R}$ that maximize the total utilitarian welfare of society.

In the second problem, the goal of the principal (perhaps the government or a monopolist) is to find the revenue-maximizing incentive-compatible assignment and transfer rules. In both problems, $x,\ t$ are taken to be measurable functions. Notice that x and t are deterministic allocation and transfer rules which is without loss of generality because utility is linear and the specific incentive compatibility constraints imposed.

The following definition of the *welfare maximization* problem takes a utilitarian efficiency criterion and uses the revelation principle in formulating the problem as a direct mechanism.

Definition: Welfare Maximization Problem

Maximize

$$\mathcal{W}(x,t) := \int_W \int_V x(v,w)vf(v)dvg(w)dw \tag{1}$$

s.t.

$$\int_W \int_V t(v,w)f(v)dvg(w)dw \geq 0 \tag{BB}$$

$$\int_W \int_V x(v,w)f(v)dvg(w)dw \leq S \tag{LS}$$

$$0 \leq x(v,w) \qquad \forall v,w \tag{NN}$$

$$t(v,w) \leq w \qquad \forall v,w \tag{BC}$$

$$\mathbb{1}_{\{w' \leq w\}}(vx(v',w') - t(v',w')) \leq vx(v,w) - t(v,w) \qquad \forall v,w,v',w' \tag{IC}$$

$$vx(v,w) - t(v,w) \geq 0 \qquad \forall v,w \tag{IR}$$

The above conditions are respectively: (BB) budget balance, (LS) limited supply, (NN) non-negative consumption, (BC) budget constraints, (IC) incentive compatibility, and (IR) individual rationality. The budget balance condition states that the principal cannot inject money into the system. If he were able to do so, then he could distribute near infinite amounts and relieve every agent's budget constraint. The non-negativity condition states that only the principal supplies the good, i.e. agents cannot be allocated a negative quantity of the good.

Before discussing the budget constraint (BC) and incentive compatibility conditions (IC), I define the *revenue maximization* problem.

Definition: Revenue Maximization Problem

Replace the welfare objective function (1) to be maximized with the following revenue function:

$$\mathcal{R}(x,t) := \int_W \int_V t(v,w)f(v)dvg(w)dw \tag{1'}$$

The revenue maximization problem has a different objective function, but retains all the constraints of the welfare maximization problem (the budget balance condition (BB) could be omitted as it would never bind).

2.2 Budget Constraints and Incentive Compatibility

The budget constraint condition (BC) states that agents cannot be asked to make a transfer $t(v,w)$ strictly larger than their wealth w. This condition differentiates the problem from an unconstrained budget setting.

The budget constraint (BC) is the same as that found in Che and Gale (2000) and Pai and Vohra (2014), but differs from that of Che, Gale and Kim (2013a), who use an ex-post constraint: $t(v,w) \leq wx(v,w)$. Dividing through the last equation by x yields $\frac{t(v,w)}{x(v,w)} \leq w$ and thus that budget constraint can also be viewed as a price per-unit constraint in the current setting. The incentive compatibility condition that I impose above states that wealthy agents *can* imitate poorer agents, but poor agents *cannot* imitate wealthier ones. This type of condition has been explained in the literature as being appropriate in a setting where agents can post bonds. In this case, agents cannot exaggerate their wealth because they are unable to post larger bonds. Another justification would be if the principal can charge a random price equal to the stated budget with non-zero probability. An alternate incentive compatibility condition is:

$$\mathbb{1}_{\{t(v',w') \leq w\}}(vx(v',w') - t(v',w')) \leq vx(v,w) - t(v,w) \qquad \forall v,w,v',w' \tag{IC'}$$

This condition is less restrictive on agents and hence more restrictive on the class of admissible mechanisms since agents can imitate any type whose transfers they can afford. In what follows, I solve for the optimal mechanism according to (IC) and find a variation of it that satisfies (IC'), which therefore will be optimal for either formulation.

Note: The incentive compatibility conditions (IC) and (IC') have both been defined in terms of a simultaneous deviation in the declaration of wealth and value. These incentive compatibility constraints could instead be expressed in terms of one-dimensional deviations as follows:

$$x(v', w) - t(v'w) \leq vx(v, w) - t(v, w) \qquad \forall v, v' \qquad \text{(Value-IC)}$$
$$\mathbb{1}_{w' \leq w}(vx(v, w') - t(v, w')) \leq vx(v, w) - t(v, w) \qquad \forall w, w' \qquad \text{(Wealth-IC)}$$

These one-dimensional ICs imply the two-dimensional IC for the following reason: If type (v, w) does not want to pretend to be (v, w') and type (v, w') does not want to pretend to be (v', w'), then type (v, w) does not want to pretend to be (v', w'), since both (v, w) and (v, w') have the same preferences over outcomes. In other words, the only determinant of an agent's preferences is his value type, while his wealth type only determines feasibility. If an agent's wealth also affects his preferences, then the two one-dimensional ICs may not imply the single two-dimensional IC constraint. If a mechanism only satisfies the value-IC constraint or the wealth-IC constraint, it will be referred to as *value-admissible* or *wealth-admissible*. If it satisfies both constraints, it will be referred to as *admissible*. Unless otherwise stated, admissibility always refers to (IC) rather than (IC').

Note: Referring back to the formulation of the welfare objective function $\mathcal{W}$, notice that it is defined without regard to aggregate transfers. The budget balance constraint requires that the principal cannot introduce money into the system, but there can be a positive net transfer of money from the agents to the principal, which reduces the agents' utility. However, for any admissible mechanism, this money could then be disbursed back to the agents in the form of a uniform lump-sum transfer without affecting any of the given constraints. Thus, any solution to the problem in which transfers are included in the welfare function will correspond to a solution of the above formulation and vice versa. As it turns out, the optimal mechanism is budget balanced, so deducting aggregate transfers from the welfare function will not change its optimality.

2.3 Assumptions

The following two assumptions will be imposed for the duration of the paper. They facilitate the subsequent analysis which will lead to a simple class of optimal mechanisms. Both concern the distribution of value types among the agents.

Assumption 1: $\frac{1-F(v)}{f(v)}$ is weakly decreasing.

Assumption 2: $f(v)$ is weakly decreasing.

Assumption 1 is a standard hazard ratio assumption. It implies that transferring the good from lower value types to higher value types will lead to an increase in revenue, since an agent's virtual valuation is increasing in his own value, as in Myerson (1981). This assumption is important because it ties together agents' valuations and their virtual valuations. Specifically, if the assumption holds, then a revenue-maximizing principal would like to deliver the good to the agents with the highest valuations, since doing so also increases the revenue generated by a given mechanism.[4]

Assumption 2 states that higher valuations are weakly less likely than lower valuations. It is satisfied by the uniform distribution, the exponential distribution, the half-normal distribution (recall that negative valuations are not possible), and any convex combination of them (as long as they have the same initial value $\underline{v}$).

Analogues of Assumptions 1 and 2 in a discrete setting are employed by Pai and Vohra (2014). Unlike Pai and Vohra (2014) and Che, Gale and Kim (2013a), there will be no further assumptions on the budget distribution aside from requiring a continuous distribution of budgets.

2.4 Interpretation and Relation to Other Models

Before proceeding to the main results, I now discuss (i) a single-agent interpretation of the model and (ii) the relationship between the model studied here and those of Pai and Vohra (2014) and Che et al. (2013a). A natural single-agent interpretation would be one where a single agent draws a valuation/wealth pair from the distributions F/G respectively. Accordingly, the welfare objective (1) expresses the ex-ante welfare criterion $\mathcal{W}(x,t) = \int_W \int_V x(v,w)vf(v)dvg(w)dw$. Additionally, the conditions (NN), (BC), (IC), (IR) naturally translate to the single-agent setting since they don't involve any aggregation. The aggregate conditions (BB) and (LS) now represent *ex-ante* conditions, i.e. budgets and supply must be balanced on average and transfers are

[4]Assumption 1 is fairly standard. However, it is actually the case that all the theorems in this analysis hold if one instead assumes the weaker regularity assumption of Myerson (1981), specifically that $v - \frac{1-F(v)}{f(v)}$ is weakly increasing, except for Part 2 of Theorem 4 of the Online Supplement. For that theorem, the stronger assumption on values is needed in order to uniquely pin down the revenue-maximizing parameters.

permitted across type realizations. Therefore, the mechanisms found in the main theorem solve the optimal mechanism design problems for the single-agent ex-ante formulation (see section 3 of Hartline (2013)).

The setting of the current model differs from that of Pai and Vohra (2014) in a number of ways. They study optimality in a single-good auction setting with either discrete types or approximate optimality in a continuous model. This enables them to follow a duality approach due to Border (1991). More fundamentally, in their model, there is a single unit of the good with an ex-post supply constraint. Formally, given any declaration of types $(v_1, w_1), \ldots, (v_N, w_N)$, the allocation x is required to satisfy $\sum_{i=1}^{N} x(v_i, w_i) \leq 1$. The limited supply constraint (LS) formulated here can be thought of as a per-person limitation. As shown in the online supplement, the finite-agent model which approaches the model studied here, has the per-person supply constraint $\sum_{i=1}^{N} \frac{x(v_i, w_i)}{N} \leq S$.

The models of Che, Gale, and Kim (2013a, 2013b) are more closely related to the current one, though they differ in a few key aspects. First, their analysis focuses on a welfare objective and generally takes place in a 2x2 (valuation,wealth) model, whereas the current paper also considers revenue and focuses on a continuum of types. In addition to an aggregate limited supply constraint, their models feature a unit demand requirement that $\forall v, w,\ x(v, w) \leq 1$. There is no such restriction in the current model, although this condition is additionally satisfied by the optimal mechanisms found here when the supply S is not too large, as demonstrated in Corollary 4. Finally, while the budget constraint of Che, Gale, and Kim (2013b) is the same as that used here, the budget constraint of Che, Gale and Kim (2013a) is fundamentally different than the one used here. There, an ex-post budget constraint is used, $t(v, w) \leq x(v, w)w \Leftrightarrow \frac{t(v,w)}{x(v,w)} \leq w$, whereas in the current paper, the budget constraint is $t(v, w) \leq w$. As discussed earlier, the difference is between a per-unit constraint and a restriction on the total amount that an agent can pay. Notice that the optimal mechanisms derived here will typically not satisfy this per-unit budget constraint.

3 The Main Result

3.1 A Preview

In this subsection, I provide a brief preview of the main result and a numerical example. The main result shows that the optimal mechanisms, from either a welfare or revenue point of view, are linear mechanisms with uniform transfers.

Definition: A *linear mechanism with uniform transfers* is characterized by two parameters, (p, T), both of which are non-negative. In these linear mechanisms, agents receive a uniform lump-sum transfer T and can purchase as much of the good as they wish at the unit price p. Agents with "low" valuations, specifically $v < p$, will therefore purchase none of the good and will simply receive the transfer T. On the other hand, agents with "high" valuations, specifically $v > p$, will have induced wealth $w + T$ and therefore will purchase $\frac{w+T}{p}$ of the good. This mechanism is clearly incentive-compatible since all agent choose their preferred bundle from the options they can afford. The mechanism is depicted in Figure 1.

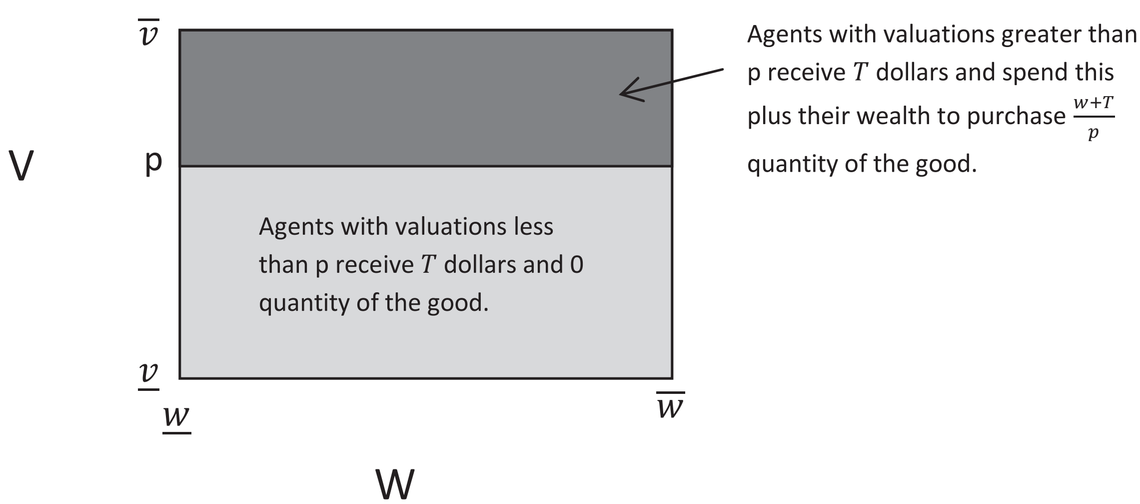

Figure 1: A Linear Mechanism with Uniform Transfers

Now, I present the main theorem, followed by three lemmas necessary for its proof:

Main Theorem: *Under Assumptions 1 & 2, the following three statements hold:*

1. *A welfare-optimal mechanism is a linear mechanism with uniform transfers,* (p_W, T_W)*, such that* $T_W = S \cdot p_W$ *and* $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$.

2. *The revenue-optimal mechanism is a linear mechanism with uniform transfers,* (p_R, T_R)*, such that* $T_R = 0$ *and* $(1 - F(p_R))\mathbb{E}[w] = S \cdot p_R$.

3. $p_W > p_R$

where the expectation $\mathbb{E}[w]$ is the average wealth of an agent in the society. Since agent's wealth and valuation are independent, this is $\mathbb{E}[w] = \int_{\underline{w}}^{\bar{w}} wg(w)dw$.

Note that the theorem demonstrates that linear mechanisms with uniform lump-sum transfers are welfare- and revenue-optimal in the class of all admissible mechanisms, not just the class of linear mechanisms. Additionally, p_W is uniquely well-defined since the defining equation $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$ has a strictly decreasing left-hand side and strictly increasing right-hand side in p_W. The same applies to p_R.

A Numerical Example: The principal has a unit supply of the good and agents' valuations and budgets are both uniformly distributed on $[0, 1]$. That is, $S = 1$ and $F = G = U(0, 1)$.

Welfare Maximization: According to the main theorem above, the welfare-maximizing price satisfies $(1 - p_W)\frac{1}{2} = p_W^2$. The unique positive solution to this equation is $p_W = \frac{1}{2}$ and each agent receives the transfer $T = \frac{1}{2}$. Agents self-select into two regimes based upon their valuations: those with valuations below 1/2 simply receive the transfer, while agents with valuations above 1/2 receive the transfer and use it along with their wealth to purchase the good. Specifically, high-value agents with original wealth 0 will have wealth 1/2 after the lump-sum transfer and will purchase one unit at the unit price of 1/2. The richest high-value agents, i.e. those with original wealth 1, will have wealth 3/2 after the uniform lump-sum transfer and therefore will purchase 3 units at the unit price of 1/2.

Revenue Maximization: Under the revenue-maximizing mechanism, there are no uniform lump-sum transfers. The market clearing price, given in the theorem is $(1 - p_R)\frac{1}{2} = p_R$ and therefore is uniquely determined as $p_R = \frac{1}{3}$. As noted in the theorem, this market-clearing price is lower than in the welfare-maximizing setting because agents receive no transfers. Since agents in the revenue-maximizing setting are "poorer", they have a lower aggregate demand function and a lower market-clearing price. Agents with valuations below 1/3 purchase none of the good. Agents with valuations above 1/3 purchase as much of the good as they can afford. Therefore, the amount purchased ranges from 0 units for high-value agents with wealth 0 to 3 units for high-value agents for agents with wealth 1. The principal's total revenue is 1/3, as opposed to 0 under the welfare-maximizing mechanism. On the other hand, the welfare of the revenue-maximizing mechanism is 2/3 as opposed to 3/4 under the welfare-maximizing mechanism.

Notice that the maximum amount purchased in the revenue-maximizing mechanism is the same as that of welfare-maximizing mechanism. This needn't be the case in general. In fact, under both mechanisms, agents fall under three possible comparisons:

1. Agents with valuations of less than 1/3 purchase none of the good in either mechanism, but receive a uniform lump-sum transfer of 1/2 in the welfare-maximizing mechanism. Therefore, they are clearly better off in the welfare-maximizing setting.

2. Agents with valuations between 1/3 and 1/2 purchase the good in the revenue-maximizing setting and receive lump-sum transfers in the welfare-maximizing setting. Of these agents, the agents whose utility increases the most in the revenue-maximizing setting are those with $v = 1/2$ and $w = 1$. They purchase 3 units for one dollar, which yields utility 3/2. On the other hand, in the welfare-maximizing setting, they have one dollar of wealth and receive another 1/2 dollar from the uniform lump-sum transfer and thus have a total utility of 3/2. Therefore, the best-off agents in the revenue-maximizing mechanism in the $[1/3, 1/2]$ valuation range are indifferent between the welfare- and revenue-maximizing mechanisms while all other agents in this valuation range strictly prefer the welfare-maximizing mechanism.

3. Agents with valuations of $v \geq 1/2$ purchase as much as they can afford in either mechanism. However, the amount they can purchase in the revenue-maximizing mechanism, i.e. $\frac{w}{1/3} = 3w$, is weakly less than the amount they can purchase in the welfare-maximizing mechanism, i.e. $\frac{w+1/2}{1/2} = 2(w + 1/2) = 2w + 1$. Therefore, they are weakly worse off in the revenue-maximizing mechanism. In fact, the inequality is strict except for those agents with $w = 1$.

In this example, the welfare-maximizing mechanism is a *Pareto improvement* over the revenue-maximizing mechanism. However, this is not a feature that needs to hold generally. More specifically, if one added a zero measure of agents with $w = 2$ and values uniformly distributed on $[0, 1]$, then agents with $w = 2$ and $v = 1$ would be worse off under the welfare-maximizing mechanism. This is because they value the good highly, but can purchase only $\frac{2+1/2}{1/2} = 5$ units in the welfare-maximizing mechanism as compared to $\frac{2}{1/3} = 6$ units in the revenue-maximizing mechanism. Therefore, while the welfare-maximizing mechanism improves utilitarian welfare relative to the revenue-maximizing mechanism, it may or may not be Pareto-improving as well.

Finally, the main result does not necessarily hold if Assumptions 1 and 2 are not satisfied. A counterexample is provided in the Appendix. The following is a brief outline of the argument used in the paper:

Step 1: I show that any admissible mechanism is simultaneously welfare- and revenue-dominated by a value-admissible mechanism where every agent receives a take-it-or-leave-it offer based upon his wealth type. I call such mechanisms *take-it-or-leave-it mechanisms.*

Step 2: I prove that any take-it-or-leave-it mechanism is welfare- and revenue-dominated by a mechanism that charges a linear price.

Step 3: I show that one can consider an equivalent linear mechanism with *uniform lump-sum transfers.*

Step 4: I find the welfare- and revenue-optimal linear mechanisms with uniform transfers.

There are a couple aspects of the derivation that are potentially troublesome and warrant further discussion. First, *take-it-or-leave-it mechanisms* are value-admissible, but may not be wealth-admissible. In fact, the only take-it-or-leave-it mechanisms that are wealth-admissible are those where wealthy agents are weakly favored. Fortunately, this does not pose a problem because in Steps 2 and 3 above, I show that an optimal take-it-or-leave-it mechanism is a linear pricing mechanism with uniform transfers and hence admissible (with respect to both (IC) and (IC')). In more detail: while I temporarily focus on take-it-or-leave-it mechanisms that are value-admissible and not necessarily wealth-admissible, the optimal take-it-or-leave-it mechanism turns out to be wealth-admissible. Therefore, the optimal take-it-or-leave-it mechanism is admissible with respect to values and budgets and therefore is the solution to the optimal mechanism design problem. These optimal mechanisms are characterized in the statement of the theorem.

3.2 Three Lemmas

The first lemma shows that if all agents have the same known budget w, then any admissible mechanism is simultaneously welfare- and revenue-dominated by a take-it-or-leave-it offer (P, Q). I use the notation P here because it is an aggregate price rather than a per-unit price. The per-unit price will be the threshold $\hat{v}$ defined in the first lemma. Moreover, the optimal take-it-or-leave-it offer has the feature that the price P equals w, the known wealth of each agent, which can be exchanged for $\hat{x}(w)$ of the good.

I show this dominance via a weight-shifting argument as outlined in Figure 3.2 where two allocation functions are drawn. The solid dark allocation function consists of a take-it-or-leave-it offer. All agents with a valuation below $\hat{v}$ receive none of the good and all agents with valuations above $\hat{v}$ receive the same amount. The light dotted allocation function is another admissible allocation function. The weight-shifting argument relies upon finding $\hat{v}$ and shifting the allocation function from the dotted allocation function to the solid one. The choice of $\hat{v}$ is uniquely determined so that the shift that takes place allocates the same supply. Thus, the good is being shifted from the region agents with valuation less than $\hat{v}$ to agents with valuation above $\hat{v}$.

Note: In the dark one-step allocation function above, it may that the implied transfer paid by the highest type, i.e. $t(\bar{v})$ is strictly less than w. In this case, $\hat{v}$

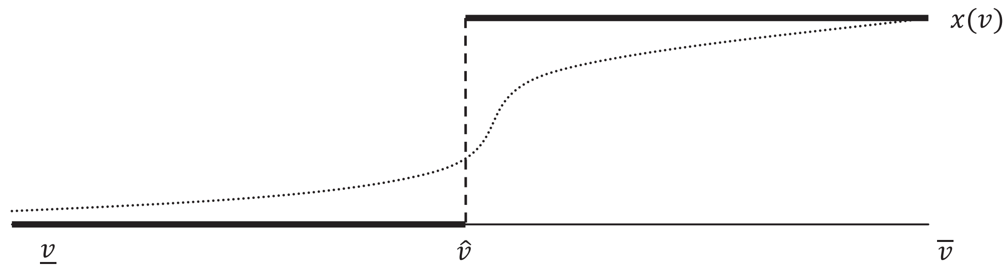

Figure 2: Agents below $\hat{v}$ no longer receive the good whereas all agents above $\hat{v}$ receive the same share as type $\bar{v}$.

(the threshold valuation/price) as well as the transferred quantity Q can both be increased such that the aggregate supply is maintained and welfare/revenue are both simultaneously improved. This step is also performed in the following lemma:

Lemma 1 *Under Assumptions 1 & 2, for a fixed level of wealth w, an admissible*[5] *allocation rule $x(v,w)$, and a transfer rule $t(v,w)$, there is a unique admissible welfare-optimal take-it-or-leave-it offer $\hat{x}$ with a transfer $\hat{t}$ that maintains the same reservation utility $U(\underline{v},w)$ and supply $\hat{S}=\int_{\underline{v}}^{\bar{v}} x(v,w)f(v)dv$. Moreover, this take-it-or-leave-it offer has the following properties:*

1. *This offer is welfare-improving.*

2. *This offer is revenue-improving.*

3. *$\hat{t}=w$.*

The lemma essentially states that any admissible mechanism is dominated by another in which an agent receives a take-it-or-leave-it offer with price equal to his wealth. It is not clear that the optimal mechanism is a take-it-or-leave-it mechanism because take-it-or-leave-it mechanisms are not necessarily admissible. Specifically, all take-it-or-leave-it mechanisms are value-admissible, but not necessarily wealth-admissible. However, the optimal take-it-or-leave-it mechanism will feature a linear

[5] The fact that we are starting off with an admissible pair of rules x, t is fundamental for the proof here.

price and will be wealth-admissible with respect to both (IC) and (IC') after a smoothing of transfers.

While the formal proof of the above lemma is relegated to the appendix, the general idea is as follows: Shifting the allocation of the good from lower- to higher-value agents is feasible as the payments induced by the envelope theorem for the highest value type weakly decrease. As agents were previously respecting their budget constraints and the highest value agent's payments weakly decrease, then he must respect his budget. This argument relies on Assumption 2. As all other agents purchase the same or less as the highest value agent, they are also asked for an affordable transfer. Shifting the good from lower-value types to higher value ones clearly improves welfare. The revenue improvement is due to the standard Assumption 1 which requires that agents' virtual valuations are increasing in their underlying value, so that the good is also being shifted to agents with higher virtual valuations.

I now turn to formally defining a take-it-or-leave-it mechanism. Notice that the definition is given in terms of cutoffs $\hat{v}(w)$ instead of quantities $\hat{x}(w)$. Either definition is equivalent via the indifference equation $\hat{x}\hat{v} - w = U(\underline{v}, w)$. These cutoffs serve as *both* the per-unit price that agents with budget w pay and the threshold for agents who buy the good.

Definition: Take-It-Or-Leave-It Mechanisms
A *take-it-or-leave-it mechanism* is a pair of functions (x, t) where $x : V \times W \to \mathbb{R}_+,\ t : V \times W \to \mathbb{R}$ satisfying the following conditions:

$$\exists \hat{v} : W \to [\underline{v}, \bar{v}] \tag{2}$$

$$\int_W \int_V t(v, w) f(v) dv g(w) dw \geq 0 \tag{3}$$

$$\int_W \int_V x(v, w) f(v) dv g(w) dw \leq S \tag{4}$$

$$\forall v, w,\ v \geq \hat{v}(w) \Rightarrow x(v, w) = \frac{w + U(\underline{v}, w)}{v},\ t(v, w) = w \tag{5}$$

$$\forall v, w,\ v < \hat{v}(w) \Rightarrow x(v, w) = 0, t(v, w) = -U(\underline{v}, w) \leq 0 \tag{6}$$

First, note that transfers are equal to an agent's wealth. It is sufficient to consider such mechanisms since Lemma 1 demonstrates that any admissible mechanism is simultaneously welfare- and revenue-dominated by such a mechanism. One can think of $\hat{v}(w)$ as being the per-unit price that an agent with wealth w faces. This price then serves as a cutoff where agents with valuations higher than the per-unit price will fully expend their budgets buying the good and agents with valuations lower than the per-unit price will not purchase any of the good and will receive a transfer. Therefore, agents' IC conditions are built into equations 5 and 6 and it is assumed that agents cannot lie about their wealth. In addition, notice that $U(\underline{v}, w) = -t(\underline{v}, w)$ and therefore equation 6 is also the individual rationality condition stating that agents who do not receive the good cannot be expected to pay anything. Take-it-or-leave-it mechanisms are a special subclass of all value-admissible mechanisms. They are not necessarily wealth-admissible because agents may wish to lie downwards about their wealth in order to obtain a cheaper per-unit price $\hat{v}(w)$.

The next lemma shows that any take-it-or-leave-it mechanism is welfare- and revenue-dominated by a linear mechanism, or in other words, one where the cutoff values $\hat{v}(w)$ are constant.

Lemma 2 *Under Assumptions 1 & 2 the welfare-optimal and the revenue-optimal mechanisms are both linear pricing systems.*

Proof: See Appendix.

These two lemmas show that any admissible mechanism is simultaneously welfare- and revenue-dominated by a linear pricing mechanism. These mechanisms are admissible according to (IC), but not necessarily for (IC') because wealthier agents may be receiving larger subsidies. The next lemma shows that for any such linear mechanism, there is a linear mechanism with lump-sum transfers that generates the same welfare and revenue. Lump-sum transfers and linear prices are (IC').

Lemma 3 *Any linear pricing mechanism that satisfies (BB), (LS), (NN), (BC), (IR), and (Value-IC) is simultaneously welfare- and revenue-equivalent to a linear mechanism with uniform lump-sum transfers.*

3.3 The Main Theorem

It remains to find the optimal linear mechanism with uniform transfers for the cases of welfare- and revenue-maximization. The theorem shows that the welfare optimum is characterized by the supply and budget balance conditions while the revenue-maximizing mechanism is characterized by the supply condition and $T = 0$. The theorem is restated below and the proof can be found in the appendix.

Main Theorem: *Under Assumptions 1 & 2, the following three statements hold:*

1. *A welfare-optimal mechanism is a linear mechanism with lump-sum transfers, (p_W, T_W), such that $T_W = S \cdot p_W$ and $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$.*

2. *The revenue-optimal mechanism is a linear mechanism with lump-sum transfers, (p_R, T_R), such that $T_R = 0$ and $(1 - F(p_R))\mathbb{E}[w] = S \cdot p_R$.*

3. $p_W > p_R$

The above theorem shows that the optimal constrained mechanism from an efficiency point of view is one where the mechanism designer charges a linear price with lump-sum transfers. Agents with high valuations will purchase as much as they can afford and agents with low valuations do not purchase the good.

In fact, the welfare-maximizing mechanism is the linear mechanism with the *highest* feasible per-unit price while the revenue-maximizing mechanism is the linear mechanism with the lowest feasible per-unit price. This is due to the inverse relationship between transfers and the market-clearing price. The welfare-maximizing principal carries out transfers that are as large as possible in order to maximally relax the agents' budget constraints, thus increasing the market-clearing price. He does so until the budget balance constraint binds him. On the other hand, the revenue-maximizing principal makes no transfers, resulting in the lowest possible market-clearing price. Consequently, the per-unit prices have the perhaps counterintuitive relationship that prices are higher in the welfare-maximizing mechanism than in the revenue-maximizing one. As this mechanism is implementable through a fixed linear price, it is dominant-strategy implementable and envy-free (in the sense that no agent envies another's ex-post allocation that he can afford; see Devanur et al. (2013)).

Mechanism Design and General Equilibrium: Typically, mechanism design and general equilibrium differ significantly in both their setup and aims. In an optimal mechanism design problem, the fundamental difficulty stems from agents' incentive compatibility constraints. In general equilibrium theory, the standard objective is Pareto optimality and welfare theorems demonstrate that the set of equilibrium allocations (with initial endowment transfers) coincide with the set of Pareto-optimal allocations. But, these transfers need not be and typically are not incentive compatible. This creates a difference in the mechanisms allowed. Specifically, mechanism design employs IC/IR mechanisms, whereas general equilibrium theory uses linear pricing mechanisms with individualized non-IC non-IR transfers. Since the optimal mechanism is a market mechanism, the optimal IC/IR mechanism happens to fall into both camps. Thus, the second welfare theorem (with the set of agents being all members of the population and the principal) implies that for the revenue-maximizing principal, there is no other allocation (whether or not it arises from an IC mechanism) that generates more revenue and Pareto improves the agents.

The welfare theorems cannot be applied to the welfare-maximizing principal because in general equilibrium agents have preferences only over their consumption and not over the other agents. For the welfare-maximizing principal, the allocation achieved in this paper is very much a second-best and there are more preferred allocations. But, these more preferred allocations, such as allocating the good only to agents with valuation type $\bar{v}$, are implementable via equilibrium only with individualized transfers, which are not incentive compatible.

Joint Objectives: The proof of the above theorem demonstrates that the optimal mechanism for a principal with an objective that depends upon on welfare and revenue is a linear pricing mechanism. The optimal linear price can be any price between p_W and p_R and the balanced budget condition makes this a one-dimensional optimization problem.

Production: In some settings, the good can be produced, perhaps at a linear cost to the principal, rather than available in finite supply. In this case, the revenue-maximizing principal sets a price above this linear cost and produces according to demand. In contrast, the welfare-maximizing principal sets a price equal to the linear cost and therefore generates no revenue with which to make transfers. Therefore, the price relationship is reversed, the welfare-maximizing principal now sets a lower price than the revenue-maximizing principal. Intuitively, if a principal can produce the good, he would rather do so than make lump-sum transfers because production is a more targeted subsidy than lump-sum transfers. For more details, see the online supplement.

Finite Markets: There is a strand of literature that focuses on computational approaches to optimal or approximately optimal mechanism design (see Yan (2011), Alaei et al. (2012) and Alaei et al. (2013); for a comprehensive survey, see Hartline (2013)). In general, these models don't apply to the current setting because the type space is infinite and because the crucial revenue-linearity property fails in a setting with private individual budgets. Nevertheless, the sequential posted-price mechanism of Yan (2011) is of particular interest. In that mechanism, there is a finite set of agents who are randomly ordered and then presented with a posted price at which they can buy or not according to this ordering. Given the current symmetric setting, this is equivalent to a posted price with rationing in case of surplus demand. It turns out that such a methodology provides a finite mechanism implementation of an approximately optimal revenue-maximizing mechanism. Yan (2011) demonstrates that such a mechanism is optimal up to a constant factor of about $(e-1)/e$. In the online supplement, I demonstrate that for any ϵ, there is an appropriate price so that with enough agents this finite mechanism with rationing is optimal within the factor $1-\epsilon$. A similar methodology also applies to the case of welfare maximization and may also be found in the supplement. Recall that Pai and Vohra (2014) address the case of a single good and therefore the interim problem with a fixed quantity of the good and a finite set of agents so far only has an approximate solution with a large number of agents.

Unit Demand: While there is no unit demand assumption, if the supply of the good and budgets are both not too large, then the characterized optimal mechanism will allocate less than one unit to every agent. In that case, allocations can be understood as probabilities and the main theorem derives the optimal mechanism for a setting with a unit demand condition as well. Formally, the unit demand condition is $x(v,w) \leq 1, \ \forall v, w$ and the following corollary holds:

Corollary 4 *If $x(\bar{v},\bar{w}) \leq 1$, then the optimal mechanisms from the main theorem are optimal for the unit demand problem as well. In terms of primitives:*

1. *For welfare optimality, the necessary and sufficient condition is* $S \leq \left(1 - F\left(\frac{\bar{w}}{1-S}\right)\right) \int_{\underline{w}}^{\bar{w}} \left(\frac{w(1-S)}{\bar{w}} + S\right) g(w) dw.$
2. *For revenue optimality, the necessary and sufficient condition is* $S \leq (1 - F(\bar{w})) \int_{\underline{w}}^{\bar{w}} \frac{w}{\bar{w}} g(w) dw.$

By the monotonicity of the optimal mechanism, the type $(\bar{v}, \bar{w})$ will receive the maximum allocation. If this allocation is less than 1, then the unit demand condition is satisfied. Therefore, the relevant condition in the welfare-maximizing case is $T_w + \bar{w}/p_W \leq 1$. Unpacking this yields $p_W \geq \bar{w}/(1-S)$. Unfortunately, this is not a straightforward condition on primitives as p_W is determined by the theorem. However, notice that part 1. of the main result defines p_W through a condition that is decreasing on the left-hand side and increasing on the right-hand side. Therefore, if the LHS $\geq$ RHS for that equation at the threshold $\bar{w}/(1-S)$, then the price p_W must be above this level. In this way, the above conditions in terms of primitives are derived.

As individual quantities sold to the highest type $(\bar{v}, \bar{w})$ are higher in the welfare-maximizing mechanism than in the revenue-maximizing mechanism, one would expect, and it indeed is the case that condition 2 is easier to satisfy than condition 1. To verify the above conditions in a specific example, the simplest case is when the wealth distribution is degenerate at w because then $w/\bar{w} = 1$ and therefore both integrals are equal to 1. Therefore, if $F \sim U[0,1]$, then condition 1 becomes $S \leq 1 - \sqrt{w}$ and condition 2 becomes $S \leq 1 - w$. Notice that for this condition to be satisfied, it must be the case that $w \leq 1$ and therefore condition 1 is tighter than condition 2.

4 Extensions

4.1 Correlation

In this subsection, I consider a setting where the revenue-maximizing mechanism may be nonlinear due to correlation between agents' valuations and budgets. Specifically, the assumption of the independence of wealths and valuations is replaced by the following weaker assumption. As mentioned in the introduction, there are settings where higher-wealth agents have higher valuations of the good (such as in the production license example).

Assumption 3: $\frac{1-F(v|w)}{f(v|w)}$ is weakly decreasing in w.

Under this assumption, I obtain the following theorem.

Theorem 5 *Under Assumptions 1-3, the revenue-maximizing mechanism is a take-it-or-leave-it mechanism with zero transfers. The optimal cutoffs $\hat{v}(w)$ are weakly decreasing in w and if Assumption 3 holds strictly, then the cutoffs are strictly decreasing whenever $\hat{v}(w) > \underline{v}$.*[6]

The proof is relegated to the appendix, but I now present the intuition behind it. As before, it is sufficient to restrict one's attention to take-it-or-leave-it mechanisms. First, it is not in the seller's interest to make any transfers to the agents since for that wealth level, he could just offer a smaller supply of the good at the same linear price. This is revenue-improving because the seller is still taking in the same revenue, selling a smaller supply which he could monetize elsewhere, and saving on the transfers to the non-purchasing agents.

The other step of the proof is to show that the optimal cutoff levels $\hat{v}(w)$ are decreasing in wealth. If this is the case, and there are no transfers to the agents, then this take-it-or-leave-it mechanism is admissible. First, any take-it-or-leave-it mechanism is value-admissible. For wealth admissibility, it is sufficient that a mechanism favors the wealthy, as this one does. Wealth admissibility then follows as wealthy types do not wish to imitate poorer types (since they are favorably treated) while poor types would like to imitate wealthy types who benefit from a cheaper per-unit price, but they are unable to do so.

Example: Suppose that $V = [0,1]$, $W = [1,100]$ and that valuations are distributed according to a truncated exponential distribution for each w. Notice that a truncated exponential distribution, $f_\lambda(v) = \lambda e^{-\lambda v}/(1 - e^{-\lambda})$ has a Hazard Rate $HR_\lambda(v) = \frac{1-F_\lambda(v)}{f_\lambda(v)} = \frac{e^{-\lambda v} - e^{-\lambda}}{\lambda e^{-\lambda v}} = \frac{1-e^{\lambda(v-1)}}{\lambda}$. This distribution satisfies both the decreasing hazard rate condition (Assumption 1) and the decreasing density condition (Assumption 2). To see that for any fixed v the hazard ratio is decreasing in λ, consider the following derivative:

$$\frac{dHR_\lambda(v)}{d\lambda} = \frac{(-(v-1)e^{\lambda(v-1)}\lambda - (1 - e^{\lambda(v-1)})}{\lambda^2} < 0$$

where the last inequality can be seen by a Taylor expansion of the exponential terms.

[6] In fact, a weaker condition would imply strict decreasingness, specifically that $\forall v$, $\frac{1-F(v,\underline{w})}{f(v,\underline{w})} > \frac{1-F(v,\bar{w})}{f(v,\bar{w})}$. This condition only requires that the hazard ratio be strictly decreasing in w at one point along every horizontal value strip rather than strictly decreasing everywhere.

Therefore, the correlated distribution $f(v,w) = f_w(v)$ satisfies Assumptions 1-3.

If the supply of the good is such that cutoffs are chosen so that $\frac{d\mathcal{R}}{dS} = 0.25$, then the value cutoffs for different types will range from about 0.25 when $w = 100$ to about 0.47 when $w = 1$. As these cutoffs are also the per-unit prices, high-wealth agents will face per-unit prices that are about half of what low-wealth agents face.

The above take-it-or-leave-it mechanism has the interesting interpretation that a decreasing cutoff function corresponds to a concave pricing function. In other words, as the quantity purchased increases, the per-unit price is decreasing. So, this is a situation where agents have fully linear utility, yet quantity discounts are profit-maximizing as a way to favor the wealthy agents. This is a form of price discrimination to capture wealthier agents with higher virtual valuations. No such price discrimination would exist in a linear utility model without budget constraints.

Remark 1: Other similar examples may be constructed by taking the exponential parameter $\lambda(w)$ to be any increasing function of w. Further examples may also be constructed by initially taking a hazard rate that satisfies the above condition, such as $(1-v)\cdot(1-w)$ on $[0,1]^2$ and deriving the underlying distribution that generates it (for details, see Thomas (1971)). For other examples of nonlinear pricing, see Wilson (1993).

Remark 2: It may be that Assumption 3 is strict for some v, but constant for the critical threshold $\hat{v}$. In this case, the optimal mechanism will be linear, despite a lack of independence.

Remark 3: The above analysis was only performed for the revenue maximizing principal because he only needs to keep track of revenue. The same analysis could be done for the welfare-maximizing principal, but it adds complication with no additional insights as the welfare-maximizing principal must keep track of both welfare and revenue (to maintain aggregate budget balance).

4.2 Aggregate Uncertainty

As shown in the online supplement, the continuum economy analyzed here is the limit of a sequence of large finite economies. Often, the continuum economy is easier to analyze because the aggregate uncertainty that exists in a finite model disappears in the limit. Nevertheless, aggregate uncertainty is possible in a continuum economy if the distribution of types $F(\Theta), G(\Theta)$ depend upon a state variable $\Theta \sim H$. We will subsequently examine how the optimal design from the main setting can be used to infer the structure of the optimal mechanisms in the case of aggregate uncertainty. Specifically, if the agent's IC/IR constraints and the principal's balanced budget/aggregate supply constraints must be satisfied state by state, then the previous analysis may be applied. One can think of this as an ex-post constraint or as an interim constraint where the agent knows both his type and the state of the world.

A key observation is that the principal can deduce Θ by asking each agent their type (v, w) he is. In a finite setting, an agent has the ability to influence the principal's estimation of the aggregate state by misreporting his type. Thus, he may have a price impact which is a feature that is not present in the continuum setting. Therefore, in the optimal mechanism design problem, the principal can, without loss of generality, treat the state of the world Θ as observable. Therefore, the constraint formulation is:

$$\int_{W(\Theta)} \int_{V(\Theta)} x(v,w,\Theta) f(v,\Theta) dv g(w,\Theta) dw \leq S \qquad \forall \Theta \qquad \text{(LS)}$$

$$\int_{W(\Theta)} \int_{V(\Theta)} t(v,w,\Theta) f(v,\Theta) dv g(w,\Theta) dw \geq 0 \qquad \forall \Theta \qquad \text{(BB)}$$

$$\mathbb{1}_{\{w' \leq w\}}(vx(v',w',\Theta) - t(v',w',\Theta) \leq vx(v,w,\Theta) - t(v,w,\Theta) \quad \forall v,w,v',w',\Theta \quad \text{(IC)}$$

$$vx(v,w,\theta) - t(v,w,\Theta) \geq 0 \qquad \forall v,w,\Theta \qquad \text{(IR)}$$

Notice that the range of budgets and valuations are now indexed by Θ as we allow the underlying wealth and valuation distributions to rely upon Θ. The optimal mechanisms satisfying the above conditions are simply linear mechanisms where the per-unit price $p(\Theta)$ and the lump-sum transfers $T(\Theta)$ are both state-dependent and defined exactly as in the main analysis. Following the prior-free implementations: the welfare-maximizing principal could, regardless of the state of the world, distribute the good and let markets clear, whereas the revenue-maximizing principal can ask for demand function reports and clear the market at the maximally clearing price.

The following question then arises: "What is the optimal mechanism where the principal is not required to satisfy the balanced budget (BB) constraint state-by-state, but rather ex-ante (BB')?"

$$\int_{\Theta}\int_{W(\Theta)}\int_{V(\Theta)} t(v,w,\Theta)f(v,\Theta)dvg(w,\Theta)dwdh \geq 0 \tag{BB'}$$

In this case, as by the main analysis, the principal state-by-state still prefers a linear mechanism with lump-sum transfers. Moreover, as the revenue-maximizing principal never makes transfers to the agents, that optimal mechanism remains the same. The innovation is that it is now possible that a welfare-maximizing principal will wish to run a surplus in some states of the world and a deficit in others because transfers have different welfare impacts in different states of the world. Thus, the welfare-maximizing principal's problem becomes to optimally choose $p_W(\Theta), T_W(\Theta)$ subject to the above constraints.

Example: Suppose that $S = 1$, $F = U(0,1)$, and $G = U(\Theta, \Theta+1)$ where Θ equals 0 or 1 with 1/2 probability each. In this example, valuations are uniform, and the aggregate state of the world determines the agents' wealth distribution.[7] As one state of the world is clearly better than the other, the notation L will refer to the "low" state where $\Theta = 0$ and similarly H will refer to the "high" state. If the principal must satisfy (BB) and cannot transfer revenue from one state to another, then the optimal mechanism is $(p_L, T_L) = (1/2, 1/2)$ and $(p_H, T_H) = \left(\frac{\sqrt{33}-3}{4}, \frac{\sqrt{33}-3}{4}\right)$. The principal has no net revenue in either state. On the other hand, if the principal can smooth across states, then his optimization problem becomes:

$$\max p_L + p_H \text{ s.t.}$$

$$(1-p_L)\left(\frac{0.5+T_L}{p_L}\right) = 1 \tag{Low State Market Clearing}$$

$$(1-p_H)\left(\frac{1.5+T_H}{p_H}\right) = 1 \tag{High State Market Clearing}$$

$$p_L + p_H - T_L - T_H \geq 0 \tag{BB'}$$

$$T_L, T_H \geq 0 \tag{IR}$$

The above equations are market-clearing conditions, a requirement for aggregate budget balance and individual rationality which imposes that the principal cannot charge a negative lump-sum transfer. The solution of the optimization problem has

[7] Θ could similarly determine the agents' valuation distributions or both simultaneously.

the feature that $p_L = p_H = \frac{\sqrt{5}-1}{2}$. The advantage of smoothing across states of the world is illustrated in the following table.

	p_L	T_L	R_L	p_H	T_H	R_H	$\mathcal{W}_L$	$\mathcal{W}_H$	$\mathcal{W}$
State-by-State Budget Balance	$\frac{1}{2}$	$\frac{1}{2}$	0	0.686	0.686	0	0.75	0.843	0.797
Across State Budget Balance	0.618	1.118	$-\frac{1}{2}$	0.618	0.118	$\frac{1}{2}$	0.809	0.809	0.809

Notice that equalizing the agents' wealth requires that the agents in the high state provide 1/2 in aggregate to insure against the low state. Therefore, welfare is reduced by about 4% in the high state and increases by about 7.9% in the low state leading to a modest overall welfare gain of about 1.6%. Prices also change significantly: decreasing about 9.9% in the high state and increasing about 23.6% in the low state.

Importantly, agents in the high state still receive a positive transfer of about 0.118, so individual rationality is satisfied in that state. If the high state was even higher so that greater transfers across states would be required, then the high-state individual rationality condition would bind and all revenue from the high state would be transferred to the low state. As this transfer would not be sufficient to equalize the agents' budgets in these two cases, the agents would still be better off in the high state than in the low one. A perhaps surprising observation is that a welfare-maximizing principal will insure agents with linear utility and no risk aversion. This can be seen more clearly seen in the main analysis, which demonstrated that the marginal utility of lump-sum transfers is declining in those transfers.

Finally, while the relative utility gain is a modest 1.6% in the above example, it can be arbitrarily large. One class of such examples is found by making the low state worse and more likely and the high state better and less likely. Similar cross-state transfers and welfare effects would be observed if the state of the world influenced valuations rather than wealth constraints.

5 Conclusion

A pair of optimal mechanism design problems were solved in a setting with a finite supply of a single divisible good and a continuum of agents with private budget constraints. It was found that the optimal mechanisms for either objective take the form of linear prices. The welfare-maximizing mechanism additionally features uniform lump-sum transfers that partially alleviate agents' budget constraints. Agents with a high valuation for the good use these transfers along with their wealth to purchase the good from the principal.

The revenue-maximizing principal offers no transfers to agents, implying that the gains in revenue from selling the good at a higher price due to the relaxation of budget constraints are more than outweighed by the cost of these transfers. An intuition of this result is that incentive compatibility requires transfers to be untargeted and thus they end up in in the hands of both agents who would use them to purchase the good and those who would not.

Additionally, two extensions were considered that introduced correlation and uncertainty respectively. I found that the revenue-maximizing mechanism when facing correlated types may feature nonlinear prices. This mechanism turns out to be a concave pricing rule and therefore offers a justification for quantity discounts, even when agents have linear utility. In the other extension, I demonstrated how to employ the main analysis to solve for the optimal mechanisms when facing aggregate uncertainty. Monetary transfers across states were welfare-improving.

To conclude, a key motivation for studying mechanism design with budget constraints is the prevalence of budget constraints, along with the fact that they may have a real impact on the structure of the optimal mechanism. Two interesting questions for future study would be to examine the case where agents have concave utility in the good or when multiple goods are available.

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Appendix

Proof of Lemma 1:

Without loss of generality, consider the case where $x(\underline{v}, w) = 0$. If this is not the case, then x can be changed so that it is so and only $t(\underline{v}, w)$ is decreased by $x(\underline{v}, w)\underline{v}$. This preserves $U(\underline{v}, w)$ the utility of an agent with value $\underline{v}$, so he still does not wish to deviate after this change. Now, an agent with value $v > \underline{v}$ is receiving utility $U(v) \geq U(\underline{v})$ and would receive $U(\underline{v})$ if he lied to be the lowest type. Therefore, she also has no incentive to deviate to pretend to be $\underline{v}$ and incentive compatibility is preserved.

For simplicity, denote $\hat{x} := x(\bar{v}, w)$. There is some $\hat{v}$ such that $(1 - F(\hat{v}))\hat{x} = \hat{S}$. In addition, let $\hat{t} = \hat{x}\hat{v} - t(\underline{v}, w)$. Now, consider the alternate transfer/allocation mechanism where

$$\hat{x}(v, w) = \begin{cases} 0 & v < \hat{v} \\ \hat{x} & v \geq \hat{v} \end{cases} \qquad \hat{t}(v, w) = \begin{cases} t(\underline{v}, w) & v < \hat{v} \\ \hat{t} & v \geq \hat{v} \end{cases}$$

It needs to be shown that this alternate mechanism is feasible, improves welfare, and improves revenue for the principal. Notice that x is weakly increasing, so $\hat{x}$ is an improvement over x because some of the good that was going to lower-valuation types under the allocation function x is being shifted to higher-valuation agents under the allocation function $\hat{x}$.

As for feasibility, it needs to be shown that $\hat{t} < w$. From the value-IC condition, transfers are given as: $t(v, w) = t(\underline{v}, w) + x(v, w)v - \int_{\underline{v}}^{v} x(z, w)dz$.

Then $t(\bar{v}, w) = t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} x(z, w)dz = t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} x(z, w)f(z)\frac{1}{f(z)}dz$ $\geq t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} \hat{x}(z, w)f(z)\frac{1}{f(z)}dz = \hat{t}(\bar{v}, w)$. In the previous calculation, notice that the inequality follows because $\frac{1}{f(z)}$ is increasing, $\int_{\underline{v}}^{\bar{v}} x(z, w)f(z)dz = \int_{\underline{v}}^{\bar{v}} \hat{x}(z, w)f(z)dz$, and because $\hat{x}(v, w) - x(v, w)$ is a weakly negative, then weakly positive, function of v.

The above argument is important, because it yields the following inequalities: $w \geq t(\bar{v}, w) \geq \hat{t}(\bar{v}, w)$ and therefore $\hat{t}$ is feasible.

Finally, it needs to be shown that

$$\mathcal{R}(\hat{t}) := \int_{\underline{v}}^{\bar{v}} \hat{t}(v, w)f(v)dv \geq \int_{\underline{v}}^{\bar{v}} t(v, w)f(v)dv$$

Using the transfer equation from above, $\mathcal{R}(t) = \int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) x(v,w)f(v)dv$. As above, we can see that $\int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) \hat{x}(v,w)f(v)dv > \int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) x(v,w)f(v)dv$ because $\frac{1-F(v)}{f(v)}$ is decreasing which implies $v - \frac{1-F(v)}{f(v)}$ is increasing, $\int_{\underline{v}}^{\bar{v}} x(z,w)f(z)dz = \int_{\underline{v}}^{\bar{v}} \hat{x}(z,w)f(z)dz$, and because $\hat{x}(v,w) - x(v,w)$ is a negative, then positive, function of v. Therefore $\mathcal{R}(\hat{t}) \geq \mathcal{R}(t)$.

Therefore the optimal transfer/allocation mechanism for the principal is a take-it-or-leave-it offer. If it is not the case that $t(\bar{v}, w) = w$, then increase $\hat{x}$ by ϵ. As before, define $\hat{v}$ s.t. $(1 - F(\hat{v}))\hat{x} = \hat{S}$ and $\hat{t} := \hat{x}\hat{v} - t(\underline{v}, w)$. Notice that $\hat{v}$ is the threshold type who accepts the take-it-or-leave-it offer, i.e. this is the amount being spent per unit. Since this has just increased, we know that revenue has increased. Moreover, we know that welfare has increased because there is now a higher threshold type. Finally, we can find an ϵ small enough s.t. $\hat{t}$ is still less than w, hence we still have feasibility. ■

Proof of Lemma 2:

Notice that for a take-it-or-leave-it mechanism, the important parameters are $x(w)$, $U(w)$, $v(w)$, and $S(w)$. These variables are respectively, the quantity of the take-it-or-leave-it offer, the reserve utility offered, the threshold value (which is also the per-unit price), and the aggregate supply that types with wealth w receive.

Focusing on one wealth strip and suppressing the function arguments, one obtains the following set of defining equations.

$$S = (1 - F(v))x \tag{7}$$

$$xv = U + w \tag{8}$$

$$\mathcal{W} = \int_{v}^{\bar{v}} zf(z)dzx \tag{9}$$

$$\mathcal{R} = (1 - F(v))(U + w) - U \tag{10}$$

I will show that $\frac{d\mathcal{W}}{dS}$ and $\frac{d\mathcal{R}}{dS}$ only depend on w via v. Then, I will show that $\mathcal{W}$ and $\mathcal{R}$ are in fact concave in S. This implies that if $v(w) < v(w')$, then there is a joint welfare and revenue improvement by removing some supply from agents with wealth w and giving it to agents with wealth w'.

Replacing x in the equations above and taking derivatives, one has

$$\frac{\partial S}{\partial v} = \frac{-vf(v) - (1-F(v))}{v^2}(U+w) \tag{11}$$
$$\frac{\partial \mathcal{W}}{\partial v} = \frac{-v^2 f(v) - \int_v^{\bar{v}} zf(z)dz}{v^2}(U+w) \tag{12}$$
$$\frac{\partial \mathcal{R}}{\partial v} = -f(v)(U+w) \tag{13}$$

Therefore,

$$\frac{d\mathcal{W}}{dS} = \frac{v^2 f(v) + \int_v^{\bar{v}} zf(z)dz}{vf(v) + (1-F(v))} \tag{14}$$
$$\frac{d\mathcal{R}}{dS} = \frac{v^2 f(v)}{vf(v) + (1-F(v))} \tag{15}$$

Therefore, both of the above derivatives are independent of agents' actual wealth levels, all that is relevant is the critical value level v. This suggests that a uniform v is optimal and this will in fact be proven once it is shown that the second derivatives are negative. However, notice that $\frac{dS}{dv} < 0$, so it will in fact suffice to show that $\frac{\partial}{\partial v}\frac{d\mathcal{W}}{dS} > 0$ and the same for $\mathcal{R}$.

The second derivatives are

$$\frac{\partial}{\partial v}\frac{d\mathcal{W}}{dS} = \frac{vf(v)(vf(v) + (1-F(v))) + vf'(v)(v(1-F(v)) - \int_v^{\bar{v}} zf(z)dz)}{(vf(v) + (1-F(v)))^2} \tag{16}$$
$$\frac{\partial}{\partial v}\frac{d\mathcal{R}}{dS} = \frac{2vf(v)(vf(v) + 1 - F(v)) + v^2 f'(v)(1-F(v))}{(vf(v) + (1-F(v)))^2} \tag{17}$$

The first term of the numerator of the welfare equation is positive. The second term is positive as well because $f'(z) \leq 0$ and $v(1-F(v)) - \int_v^{\bar{v}} zf(z)dz \leq v(1 - F(v)) - \int_v^{\bar{v}} vf(z)dz = 0$.

As for the revenue equation, notice that $\frac{1-F(v)}{f(v)}$ decreasing and this implies that $f'(v)(1-F(v)) \geq -f(v)^2$. Therefore, the numerator of the revenue equation is bounded below by $v^2 f(v)^2 + 2vf(v)(1-F(v)) \geq 0$. ■

Proof of Lemma 3:

From the previous argument, one sees that the optimal mechanism is a linear-pricing system p. However, if non-uniform transfers are being made to the agents with lowest valuations $\underline{v}$, then this mechanism is not admissible with respect to (IC'). This is because, agents who do not wish to purchase the good may wish to misreport their types in order to secure a more favorable transfer. Formally, admissibility with respect to (IC') will fail if for $w \neq w'$, it is the case that $t(w, v) \neq t(w', v)$.

This problem is easily solved by distributing a uniform transfer equal to $\int_{\underline{w}}^{\bar{w}} t(w, \underline{v}) g(w) dw$ to agents who do not purchase the good. This may change the allocation for many agents (not just the ones who do not purchase the good), but notice that this change is welfare and revenue equivalent.[8]

Formally, the quantity demanded at the price p does not change:

$$
\begin{aligned}
&(1 - F(p)) \int_{\underline{w}}^{\bar{w}} \frac{w + t(\underline{v}, w)}{p} g(w) dw \\
=& \frac{1 - F(p)}{p} \left(\int_{\underline{w}}^{\bar{w}} w g(w) dw + \int_{\underline{w}}^{\bar{w}} t(\underline{v}, w) g(w) dw \right) \\
=& \frac{1 - F(p)}{f(p)} \int_{\underline{w}}^{\bar{w}} \left(w + \int_{\underline{w}}^{\bar{w}} t(\underline{v}, z) g(z) dz \right) g(w) dw
\end{aligned}
$$

where the first line is the amount demanded with non-uniform transfers and the last line is the quantity demanded with uniform transfers. This implies that the market clearing price p does not change. Since the quantity supplied does not change, and the market clearing price does not change, neither does welfare or revenue. This is because for any v above p, the same quantity is being bought and the same transfers are being made for that horizontal slice of agents. ■

Proof of Main Theorem:

Suppose that the linear mechanism with uniform transfers (p, T) does not supply the entire supply S of the good. Then the price p can be slightly reduced and this increases both the welfare of the agents (because every agent is receiving weakly more)

[8] Recall that agents were using their wealth w and transfer $t(w, \underline{v})$ to buy the good at price p, so agents who were receiving favorable transfers could purchase more of the good. Smoothing the transfers therefore affects purchasers because their effective budget changes.

and increases revenue (because a larger set of agents is paying their entire wealth). Therefore, for either optimal mechanism, it must be that the supply constraint binds: $(1 - F(p))\frac{\mathbb{E}[w]+T}{p} = S$.

Welfare-Maximization:

For the case of welfare-maximization, suppose that there is some leftover revenue. Then the principal could increase the transfers T slightly and increase the price p slightly and improve the overall utilitarian welfare. Therefore, the welfare-maximizing mechanism needs to satisfy the budget balance constraint, specifically: $(1 - F(p))\mathbb{E}[w] = F(p)T$.

Solving for p and T yields the conditions provided in the theorem.

Revenue-Maximization:

For the case of revenue-maximization, suppose that $T > 0$. Then, an alternate mechanism where p is unchanged, and T is changed to 0 generates strictly more welfare. Therefore, the revenue-maximizing mechanism (p, T) is such that $T = 0$ and the supply constraint above binds.

Substituting in for T and multiplying the supply equation by p yields the conditions provided in the theorem. ■

Proof of Theorem 5:

Notice that the cutoff functions are determined by Equation 15, restated here in the case of non-independence of values and budgets.

$$\frac{d\mathcal{R}}{dS} = \frac{v^2 f(v|w)}{vf(v|w) + (1 - F(v|w))}$$

Now, because values and budgets are not independent, there may be differential returns to transferring supply from some wealth levels to other wealth levels. It needs to be checked that these returns are increasing in wealth. Technically, it is necessary and sufficient that the above condition is increasing in wealth. Dividing the top and bottom of the equation through by $f(v|w)$ yields:

$$\frac{d\mathcal{R}}{dS} = \frac{v^2}{v + \frac{1-F(v|w)}{f(v|w)}} \tag{18}$$

By Assumption 3, the last term of the denominator is weakly decreasing in w, and hence, the whole expression is weakly increasing in w. In addition, the derivative is always positive. Therefore, the revenue-maximizing mechanism is where all S of the good is being sold, and where $\frac{d\mathcal{R}}{dS}$ is constant for every w being sold the good. This is precisely what is desired, and completes the proof. ■

A Nonexample

Here, I provide an example where the welfare maximizing mechanism is not a linear mechanism. This will be a "counterexample" to the consequence of my theorem in the sense that my theorem points to the optimality of linear mechanisms. Of course, it is not a "counterexample" to the theorem as I consider a situation where the antecedents of the theorem do not hold. Specifically, the assumptions on agents' valuations that I make in the body of the paper will not hold in the following example.

Recall that, by a linear mechanism, I mean that there is a per-unit price p and to purchase a quantity of the good x, the price is px. In addition, my main result of the optimality of linear mechanisms applies to a either a single uniform budget constraint or individual unobservable budget constraints.

Therefore, for simplicity, I will show a "counterexample" in the single uniform budget setting where a linear mechanism is not optimal. In this case, a linear mechanism has the additional feature that it is a take-it-or-leave-it mechanism. Specifically, any linear mechanism has a threshold value. Above this threshold, every agent receives the same quantity for the same transfer, and below this threshold, no agent purchases the good.

I prove the non-optimality of a take-it-or-leave-it mechanism in this setting by demonstrating a mechanism that dominates it.

Consider the following value distribution:

$$f(x) = \begin{cases} 1/12 & \text{if } 1 \leq x \leq 2 \\ 1/3 & \text{if } 2 < x \leq 3 \\ 7/12 & \text{if } 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

The above distribution has three steps from 1 to 4. Suppose that $S = 11/12$ and $w = 2$ and consider a mechanism that sells one unit of the good at the price 2. This is a linear price mechanism for a single wealth level and will generate welfare:[9]

$$\mathcal{W}_1 = \frac{1}{3} \cdot \frac{5}{2} + \frac{7}{12} \cdot \frac{7}{2} = \frac{23}{12} = 1.9333$$

On the other hand, consider a mechanism where the principal sells the good according to:

$$x(v) = \begin{cases} 19/29 & \text{if } 1 \leq x \leq 3 \\ 32/29 & \text{if } 3 < x \leq 4 \end{cases} \qquad t(v) = \begin{cases} 19/29 & \text{if } 1 \leq x \leq 3 \\ 2 & \text{if } 3 < x \leq 4 \end{cases}$$

The above I call a two-step mechanism since there are two different possible allocations and the above mechanism generates welfare

$$\mathcal{W}_2 = \frac{19}{29} \cdot \frac{1}{12} \cdot \frac{3}{2} + \frac{19}{29} \cdot \frac{1}{3} \cdot \frac{5}{2} + \frac{32}{29} \cdot \frac{7}{12} \cdot \frac{7}{2} = 2.881$$

The two-step mechanism generates less revenue and more welfare than the one-step mechanism. The additional revenue from the one-step mechanism can be redistributed to the population, so that its welfare can be improved by finding a higher linear price at which trade can take place. Doing so would then make the welfare comparison between the two above mechanisms unclear. However, such a redistributional improvement can be prevented by adding a large population of agents with valuations between 0 and 1. These agents will then absorb most of the cash distributions. I show how this is done in the subsequent paragraphs.

Define a measure g as:

$$g(x) = \begin{cases} m & \text{if } 0 \leq x < 1 \\ f(x) & \text{otherwise} \end{cases}$$

The above is a measure and not a density because the integral of g is equal to $m + 1$ and not 1. Now, I will compare two mechanisms based upon the previously defined ones. The first is the one-step mechanism as defined before, with all of the money taken in, redistributed to the population so that a higher per-unit price can be established. This is the welfare-maximizing one-step mechanism in the class of all one-step mechanisms. The other is the two-step mechanism from before with no cash

[9]The welfare calculation is performed by looking at each of the populations and multiplying quantity (=1) times density (= 1/3 or 7/12) times average value (= 5/2 or 7/2).

distributions. Therefore, it is clear that $0 = \mathcal{R}_1(m) < \mathcal{R}_2(m)$.

The important point is that as $m \to \infty$ the additional welfare value of the extra revenue that the one-step function generates converges to 0. Specifically, consider if all money that is received is redistributed, then, the one-step mechanism implies a cutoff price p where $(1 - F(p))w = Sp\frac{F(p)+m}{1+m}$. Moreover, recall that $(1 - F(2))w = S \cdot 2$. So, as $m \to \infty$, it is the case that $\frac{F(p)+m}{1+m} \to 1$ and therefore the threshold price $p \to 2$. This implies $\mathcal{W}_1(m) \to \mathcal{W}_1 < \mathcal{W}_2 = \mathcal{W}_2(m)$.

So, for large enough m, it is the case that $\mathcal{W}_1(m) < \mathcal{W}_2(m)$. While g is not a distribution, one can consider a rescaling that is, specifically, let $h(x) = \frac{g(x)}{m+1}$ and supply be $\frac{S}{m+1}$. Then the above allocation mechanisms are still applicable, but generate welfare $\frac{\mathcal{W}_1(m)}{m+1}$ and $\frac{\mathcal{W}_2(m)}{m+1}$ respectively. Therefore, it is the case that the 2-step mechanism generates strictly higher revenue and welfare then the 1-step mechanism for the distribution h and supply $\frac{S}{m+1}$. ■